Pre-class Warm-up!!!

Question: True or False, for vectors v_1, … , v_6 in R^n ?

True False

a. If v_1, … , v_6 are linearly independent then v_1, … , v_4 are necessarily linearly independent.

True

b. If v_1, … , v_4 are lin. indep. then v_1, … , v_6 are necessarily lin. indep.

False

$$\underbrace{\overbrace{(1\,0\,0\,0)\ (0\,1\,0\,0)\ (0\,0\,1\,0)\ (0\,0\,0\,1)}^{\text{independent}}\ (1\,0\,0\,0)\ (0\,1\,0\,0)}_{\text{dependent}}$$

c. If v_1, … , v_6 span R^n then v_1, … , v_4 necessarily span R^n.

False

d. If v_1, … , v_4 span R^n then v_1, … , v_6 necessarily span R^n.

True. If $b = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$

then $b = \text{same} + 0v_5 + 0v_6$.

Section 4.4: Bases and dimension

We learn:

- The meaning of the word basis, and a broader definition of the word dimension.
- Theorem: Any two bases for a vector space have the same size.
- Theorem: A basis is a maximal independent set, and also a minimal spanning set.
- How to find a basis for various vector spaces: the solution set to a homogeneous system of equations, lines and planes in R^3.

Definition on page 235:
A set of vectors S = {v_1, … ,v_k} is a basis for a vector space V if and only if
a. the vectors are independent, and
b. they span V.

Example. (1, 0, -1), (1, 2, -2), (1, 6, -3)

We have already seen these are linearly independent and span $\mathbb{R}^3$

(Reduce $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ -1 & -2 & -3 \end{bmatrix}$ to reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. etc.)

These vectors are a basis for $\mathbb{R}^3$

Equivalently: S is a basis for V <=>

every vector b in V can be written (span) uniquely (independent) as a linear combination of the vectors in S,

If unique, then the 0 vector is uniquely a linear combination of vectors in S so they are independent.

If $a_1v_1 + \cdots + a_kv_k = b_1v_1 + \cdots + b_kv_k = w$

then $(a_1 - b_1)v_1 + \cdots + (a_k - b_k)v_k = w - w = 0$

independent $\Rightarrow a_i - b_i = 0$ always

$a_i = b_i$.

Example: the standard basis for R^3 is

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. It is a basis

Like question 12 - 14:
Find a basis for the plane in R^3 with equation $2x - y + 3z = 0$.

Solution: The plane is the solution set to

$$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$ is in echelon form. y, z are free variables. The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(y-3z) \\ y \\ z \end{bmatrix} = y\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Do these vectors span, and are they independent?

Claim $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$ is a basis for this plane.

Span ✓ Are they independent?

If $y\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} ? \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then

$y = 0$ and $z = 0$, so they are independent.

Like questions 15 - 26:
Find a basis for the solution space of the system

$$\begin{aligned} w + 2x + 3y + 4z &= 0 \\ 2w + 4x + 7y + 9z &= 0 \\ 3w + 6x + 9y + 12z &= 0 \end{aligned}$$

Question: determine whether the vectors (1, 0, -1), (1, 2, -2), (1, 6, -3) form a basis for R^3.

Solution: Reduce $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ -1 & -2 & -3 \end{bmatrix}$

to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The vectors are independent and span (say more) as before.

Further question:

a. Does the vector (1,6,-3) lie in the span of the vectors (1, 0, -1) and (1, 2, -2)?

Yes No ✓

b. Does the vector (1,2,-2) lie in the span of the vectors (1, 0, -1) and (1, 6 -3)?

Yes No ✓

Theorem 1: If {v_1, … ,v_n} is a basis for V then any set of vectors w_1, … ,w_r with r > n is dependent.

Proof. Write each $w_j = \sum_{i=1}^{n} a_{ij} v_i$, $j = 1, \dots, r$

Make a matrix

$A = (a_{ij}) =$ is an $n \times r$ matrix $r > n$.

It has a free variable, so a non-zero solution $\underline{x} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$ to $A \underline{x} = \underline{0}$

Now for each i, $\sum_{j=1}^{r} \lambda_j a_{ij} = 0$

Multiply each equation by v_i and add

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{r} \lambda_j a_{ij} \right) v_i = \underline{0} = \sum_{j=1}^{r} \lambda_j \left(\sum_{i}^{n} a_{ij} v_i \right)$$

$$= \sum_{j=1}^{r} \lambda_j w_j$$

This finds a non-zero linear combination of the w_j that equals $\underline{0}$. The w_j are dependent.

Theorem 2: Any two bases for a vector space have the same size.

Proof. If S and T are bases for V and S has more vectors than T then the vectors in S must be dependent by Thm. 1, so S couldn't be a basis. Thus S and T have the same size.

Definition. The dimension of a vector space is the size of a basis.

Pre-class Warm-up!!!

Question: have we already proved the following theorem?:

Theorem
Let V be a vector space of dimension n and let S be a set of n vectors in V.
If S is linearly independent then S spans V and hence is a basis for V.

Yes No

What about:

Theorem
Let S be a set of n vectors in R^n.
If S is linearly independent then S spans V and hence is a basis for V.

$\mathbb{R}^n$ $\mathbb{R}^n$

Yes No

Theorem 3 (second 1/2 of it)

(c) If S is a set of linearly independent vectors in a vector space V then S is contained in a basis for V. S can be extended to a basis for V.

(d) If S is a set of vectors that spans V then S contains a basis for V.

(c) is in the homework (questions 29 and 30).
(d) is questions 31 and 32.

(c) implies that every vector space has a basis.

(d). Assume S is a finite set of vectors

If they are independent, they are a basis, done. Otherwise some vector can be written as a linear combination of the others. Remove it. The span of the smaller set is also V. Repeat until we get an independent set; a basis!

Theorem 3 (first 1/2 of it)

Let V be a vector space of dimension n and let S be a set of n vectors in V.

(a) If S is linearly independent then S is a basis for V.

(b) If S spans V then S is a basis for V.

Proof (a). S is lin. ind. so is contained in a basis by (c). This basis has n vectors. S is the basis.

(b) Similar.

Example (like example 4):

Let V be the set of polynomials

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

(a) Show that V has dimension 4.

(b) Show that 1, 1+x, x+x^2, x^2+x^3 is a basis for V